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THE CONDITION FOR AN ISOTHERMAL FAMILY ON A SURFACE.

BY JAMES K. WHITTEMORE.

Consider a real surface and let the rectangular coördinates of its points be given as functions of the two real parameters u, v; suppose the linear element given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

The condition that a family of curves on the surface, $\lambda(u, v) = c$, be isothermal, as generally given, is that $\Delta_2(\lambda)/\Delta_1(\lambda)$ be a function of λ, where $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ are the first and second differential parameters formed with respect to the linear element of the surface.* This condition is not applicable in the case of frequent occurrence where the family of curves is given not in finite form but by a differential equation. Lie has proved† that if u, v are isothermic parameters in a plane the integral curves of the differential equation $dv/du = \alpha(u, v)$ form an isothermal family when and only when arc tan α is a harmonic function of u and v, and that in this case the equation may be integrated by quadratures. It may be remarked that the theorem is given by Lie as an application of his method of solving a differential equation admitting a known infinitesimal transformation; further, that his proof applies without change to the case of any surface given in terms of isothermic parameters. Lie has also shown‡ that the equation can be integrated by two quadratures if it defines an isothermal family on any surface given with any coördinates, but he has given no method of determining when this is the case.

In this paper we obtain by a simple method, quite different from Lie's, the condition that the differential equation, $dv/du = \alpha$, define an isothermal family on any real surface given with any real parameters u, v, a condition which is a generalization of Lie's condition for isothermic parameters; we prove Lie's theorem that the equation can be integrated by two quadratures when it defines an isothermal family; finally we give the geometrical significance of the angles of the complex integrating factors of the differential equations of the minimal lines of the surface.

Let ω be the angle measured from the positive direction of (v) to the positive direction of (u), where (u) and (v) mean the curves u constant and v constant respectively. Then

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* See Eisenhart, Differential Geometry (1909), pp. 84, 89, 96.

† Lie-Scheffers, Differentialgleichungen (1891), pp. 156, 157.

‡ L. c., pp. 160–162.

$$\sin \omega = \frac{H}{\sqrt{EG}}, \qquad \cos \omega = \frac{F}{\sqrt{EG}},$$

where $\sqrt{EG}$ is positive and H is the positive square root of $EG - F^2$. Let φ be the angle measured in the same direction as ω from the positive direction of (v) to the positive direction on that integral curve C of $dv/du = \alpha$ which passes through the point u, v. The positive directions on (v) and on C are the directions in which the parameter u increases. Considering the infinitesimal triangle whose sides are (v), $(u + du)$, C, we have

$$\frac{\sqrt{G}\,dv}{\sqrt{E}\,du} = \frac{\sqrt{G}}{\sqrt{E}}\alpha = \frac{\sin \varphi}{\sin(\omega - \varphi)} = \frac{\sqrt{EG}}{H \cot \varphi - F},$$

from which

$$\alpha = \frac{E}{H \cot \varphi - F}, \qquad \tan \varphi = \frac{H\alpha}{E + F\alpha}. \tag{1}$$

The minimal lines of the surface are given by $ds^2 = 0$. Since

$$E ds^2 = (Edu + Fdv)^2 + H^2 dv^2$$

the minimal lines are the integral curves of the two equations,

$$Edu + (F - iH)dv = 0, \qquad Edu + (F + iH)dv = 0. \tag{2}$$

We may assume since E, F, H, u, v are real that integrating factors of equations (2) are respectively $\rho e^{i\theta}$ and $\rho e^{-i\theta}$. Then

$$\begin{aligned} \rho e^{i\theta}[Edu + (F - iH)dv] &= dx + idy, \\ \rho e^{-i\theta}[Edu + (F + iH)dv] &= dx - idy. \end{aligned} \tag{3}$$

Since these equations give

$$ds^2 = \frac{1}{\rho^2 E}(dx^2 + dy^2)$$

x and y are isothermic parameters of the surface, and are, with a suitable choice of ρ, θ, any pair of isothermic parameters of the surface. Either of equations (3) gives

$$\begin{aligned} dx &= \rho[(Edu + Fdv)\cos\theta + H \sin\theta dv], \\ dy &= \rho[(Edu + Fdv)\sin\theta - H \cos\theta dv]. \end{aligned} \tag{4}$$

The conditions of integrability for equations (4) are

$$\frac{\partial}{\partial v}(\rho E \cos\theta) = \frac{\partial}{\partial u}(\rho F \cos\theta + \rho H \sin\theta),$$

$$\frac{\partial}{\partial v}(\rho E \sin\theta) = \frac{\partial}{\partial u}(\rho F \sin\theta - \rho H \cos\theta).$$

Expanding and combining the last two equations, we have

$$(5)\qquad \frac{\rho_u}{\rho} H + H_u + E\theta_v - F\theta_u = 0,$$

$$\frac{\rho_v}{\rho} EH + H(E_v - F_u - H\theta_u) + F(H_u + E\theta_v - F\theta_u) = 0,$$

where subscripts denote partial differentiation. The condition of integrability of (5), considered as equations in ρ, is

$$\frac{\partial}{\partial v}\left[\frac{H_u + E\theta_v - F\theta_u}{H}\right] = \frac{\partial}{\partial u}\left[\frac{H(E_v - F_u - H\theta_u) + F(H_u + E\theta_v - F\theta_u)}{EH}\right].$$

The last equation may be reduced to

$$(6)\qquad \Delta_2\theta = \frac{1}{H}\frac{\partial}{\partial u}\left[\frac{\partial}{\partial v}\left(\log\frac{E}{H}\right) - \frac{H}{E}\frac{\partial}{\partial u}\left(\frac{F}{H}\right)\right].$$

If θ is a solution of (6), ρ is found from (5) by a quadrature, then y from (4) by a quadrature. The equation $y = c$ gives an isothermal family and is the general solution of $dy = 0$ or

$$\frac{dv}{du} = \frac{E}{H\cot\theta - F}.$$

Comparing the last equation with (1) it appears that the necessary and sufficient condition that the differential equation, $dv/du = \alpha$, define an isothermal family is that the angle φ measured from (v) to the integral curve C and equal to

$$\operatorname{arc\,tan}\frac{H\alpha}{E + F\alpha}$$

be a solution θ of (6). The angles of the complex integrating factors of the two differential equations of the minimal lines (2) are plus and minus the angle of intersection of the curves of an isothermal family with the curves (v).

When u, v are isothermic parameters, the condition given is that of Lie, for equations (1) and (6) become

$$\tan\varphi = \alpha, \qquad \frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = 0.$$

We remark that if θ_1 and θ_2 are two solutions of (6), then $\Delta_2(\theta_1 - \theta_2) = 0$, that is, the angle of intersection ψ of two isothermal families is such that

$\Delta_2\psi = 0$, in particular a harmonic function of any pair of isothermic parameters. It may also be easily proved that when u, v are isothermic parameters the necessary and sufficient condition that the equation,

$$du^2 - dv^2 + 2 \tan \varphi du dv = 0,$$

define an isothermal system is that φ be harmonic.

NEW HAVEN,
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